**MATH2050B 1920 HW5**
TA's solutions[1] to selected problems

**Q1.** Suppose $f = g$ on $V_\delta(c) \cap (A \setminus \{c\})$ for some $\delta > 0$, show that $\lim_{x\to c} f(x) = l$ iff $\lim_{x\to c} g(x) = l$.

**Solution.** ($\Rightarrow$) Suppose that $\lim_{x\to c} f(x) = l$. Let $\epsilon > 0$, then there is $\eta > 0$ s.t. for all $x \in V_\eta(c) \setminus \{c\}$,

$$|f(x) - l| < \epsilon.$$

Let $\delta' = \min(\delta, \eta)$. Then $\delta' > 0$, and for all $x \in V_{\delta'}(c) \setminus \{c\}$, $f(x) = g(x)$, so

$$|g(x) - l| < \epsilon.$$

Hence $\lim_{x\to c} g(x) = l$.

($\Leftarrow$) Same as above.

**Q2.** Show, by def, that $\lim_{x\to c} f(x) = l$ iff $\lim_{x\to c}(f(x) - l) = 0$.

**Solution.** ($\Rightarrow$) Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = l$, so there is $\delta > 0$ s.t. for all $x \in V_\delta(c) \setminus \{c\}$,

$$|(f(x) - l) - 0| = |f(x) - l| < \epsilon.$$

Hence $\lim_{x\to c}(f(x) - l) = 0$.

($\Leftarrow$) If $\lim_{x\to c}(f(x) - l) = 0$. Let $\epsilon > 0$. Then there is $\delta > 0$ s.t. for all $x \in V_\delta(c) \setminus \{c\}$,

$$|f(x) - l| = |f(x) - l - 0| < \epsilon.$$

Hence $\lim_{x\to c} f(x) = l$.

**Q3.** Show, by def, that $\lim_{x\to c} f(x) = l$ iff $\lim_{x\to 0} f(x + c) = l$.

**Solution.** ($\Rightarrow$) Let $\epsilon > 0$. Then there is $\delta > 0$ s.t. for all $y \in V_\delta(c) \setminus \{c\}$,

$$|f(y) - l| < \epsilon.$$

Therefore, for all $x \in V_\delta(0) \setminus \{0\}$, we have $x + c \in V_\delta(0) \setminus \{0\}$, so

$$|f(x + c) - l| < \epsilon.$$

Hence $\lim_{x\to 0} f(x + c) = l$.

($\Leftarrow$) Let $\epsilon > 0$. Then there is $\delta > 0$ s.t. for all $y \in V_\delta(0) \setminus \{0\}$,

$$|f(y + c) - l| < \epsilon.$$

Therefore, for all $x \in V_\delta(c) \setminus \{c\}$, we have $x - c \in V_\delta(0) \setminus \{c\}$ and so

$$|f(x) - l| = |f(x - c + c) - l| < \epsilon.$$

Hence $\lim_{x\to c} f(x) = l$.

**Q4.** For $A = \mathbb{R}$, show by definition, that $\lim_{x\to 0} f(x) = l$ iff $\lim_{x\to 0} f(100x) = l$.

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

**Solution.** ($\Rightarrow$) Let $\epsilon > 0$. Then there is $\delta > 0$ s.t. for all $y \in V_\delta(0) \setminus \{0\}$,

$$|f(y) - l| < \epsilon.$$

Put $\delta' = \delta/100$. Then for all $x \in V'_\delta(0) \setminus \{0\}$, we have $100x \in V_\delta(0) \setminus \{0\}$ and so

$$|f(100x) - l| < \epsilon.$$

Hence $\lim_{x\to 0} f(100x) = l$.

($\Leftarrow$) Let $\epsilon > 0$. Then there is $\delta > 0$ s.t. for all $y \in V_\delta(0) \setminus \{0\}$,

$$|f(100y) - l| < \epsilon.$$

Put $\delta' = 100\delta$. Then for all $x \in V'_\delta(0) \setminus \{0\}$, we have $x/100 \in V_\delta(0) \setminus \{0\}$ and so

$$|f(x) - l| = |f(100(\frac{x}{100})) - l| < \epsilon.$$

Hence $\lim_{x\to 0} f(x) = l$.

**Q5.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \,, \text{ if } x \in \mathbb{Q} \\ 3 \,, \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\lim_{x\to c} f(x)$ exists in $\mathbb{R}$ iff $c = 3$.

**Solution.** We prove that $\lim_{x\to 3} f(x) = 3$ and $\lim_{x\to c} f(x)$ does not exist for $c \neq 3$.

Let $\epsilon > 0$. Take $\delta = \epsilon$. For all $x \in V_\delta(3) \setminus \{3\}$,

**Case 1.** $x \in \mathbb{Q}$, then $f(x) = x$ and $|f(x) - 3| = |x - 3| < \delta = \epsilon$.

**Case 2.** $x \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x) = 3$, then $|f(x) - 3| = 0 < \epsilon$.

Hence $\lim_{x\to 3} f(x) = 3$.

For $c \neq 3$, there is $\epsilon > 0$ so that $|c - 3| > \epsilon > 0$. Choose a sequence $(x_n)$, $x_n \in \mathbb{Q}$, $x_n \to c$. Choose another sequence $(y_n)$, $y_n \in \mathbb{R} \setminus \mathbb{Q}$, $y_n \to c$. Then $f(x_n) = x_n \to c$, $f(y_n) = 3 \to 3$. This shows that $f$ does not have a limit as $x \to c$.

**Q6.** Suppose $\lim_{x\to c}(f(x))^2 = l \geq 0$. Can we conclude that $\lim_{x\to c} f(x) = \sqrt{l}$? (Yes if $l = 0$ but not otherwise, any counter example?)

**Solution.** For $l = 0$. We show that $\lim_{x\to c} f(x) = 0$. Let $\epsilon > 0$. Then $\epsilon^2 > 0$. Since $\lim_{x\to c}(f(x))^2 = 0$, so there is $\delta > 0$, s.t. for all $x \in V_\delta(c) \setminus \{c\}$,

$$|f(x)^2| < \epsilon^2.$$

Hence for all $x \in V_\delta(c) \setminus \{c\}$,

$$|f(x)| < \epsilon.$$

$\therefore \lim_{x\to c} f(x) = 0$.

If $l > 0$. Take $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{l} & , \text{ if } x \in \mathbb{Q} \\ -\sqrt{l} & , \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then $f^2 \equiv l$ on $\mathbb{R}$, so for all $c \in \mathbb{R}$, $\lim_{x\to c} f(x)^2 = l$. But $\lim_{x\to c} f(x)$ does not exist for all $c \in \mathbb{R}$. (Reason: take a rational sequence $(x_n)$, and an irrational sequence $(y_n)$, both converge to $c$, but $f(x_n) \to \sqrt{l}$, $f(y_n) \to -\sqrt{l}$)

**Q7.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x+2\,, & \text{if } x \in \mathbb{Q} \\ 3x-1\,, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Exactly at what $c$ such that $\lim_{x\to c} f(x)$ exists? And what is the limit then?

**Solution.** $\lim_{x\to c} f(x)$ exists iff $c = 3/2$.

Let $f_1(x) = x + 2$, $f_2(x) = 3x - 1$ for $x \in \mathbb{R}$. Then $\lim_{x\to 3/2} f_1(x) = \lim_{x\to 3/2} f_2(x) = 7/2$. Therefore, for any $\epsilon > 0$, there is $\delta > 0$ s.t. for all $x \in V_\delta(3/2) \setminus \{3/2\}$,

$$|f_1(x) - \frac{7}{2}| < \epsilon\,, \qquad |f_2(x) - \frac{7}{2}| < \epsilon.$$

Then, for all $x \in V_\delta(3/2) \setminus \{3/2\}$, we have that $f(x)$ is either $f_1(x)$ or $f_2(x)$. In any case, $|f(x) - 7/2| < \epsilon$. Hence $\lim_{x\to 3/2} f(x) = 7/2$.

For any other $c$, $\lim_{x\to c} f_1(x) \neq \lim_{x\to c} f_2(x)$. So $\lim_{x\to c} f(x)$ cannot exist. Reason: choose a rational sequence $(x_n)$, $x_n \to c$; choose an irrational sequence $(y_n)$, $y_n \to c$. Then

$$\lim_n f(x_n) = \lim_n f_1(x_n) = \lim_{x\to c} f_1(x) \neq \lim_{x\to c} f_2(x) = \lim_n f_2(y_n) = \lim_n f(y_n)$$

**Q8.** Find $\delta > 0$ s.t. on $V_{\delta_i}(1)$

$$|x^2 - 1| < \epsilon_i$$

where $\epsilon_1 = \frac{1}{2}$, $\epsilon_2 = \frac{1}{10}$ and $\epsilon_3 = \frac{1}{100}$.

**Solution.** It only needs to find $\delta_3$(WHY?). We determine $\delta_3$ later, and first let us notice if $0 < \delta < 1$ and $x \in V_\delta(1)$, i.e. $1 - \delta < x < 1 + \delta$, then

$$(1-\delta)^2 < x^2 < (1+\delta)^2,$$

so

$$(1-\delta)^2 - 1 < x^2 - 1 < (1+\delta)^2 - 1,$$

that is,

$$-2\delta + \delta^2 < x^2 - 1 < 2\delta + \delta^2.$$

We want $-\frac{1}{100} < -2\delta + \delta^2$ and $2\delta + \delta^2 < \frac{1}{100}$. Solving the first inequality gives $\delta > 1 + \frac{3}{10}\sqrt{11}$ or $\delta < 1 - \frac{3}{10}\sqrt{11} \approx 0.00501\ldots$. Solving the second gives $-1 - \frac{\sqrt{101}}{10} < \delta < -1 + \frac{\sqrt{101}}{10} \approx 0.00498\ldots$. At the same time we need $\delta > 0$. Hence any $\delta_3 \in (0, 0.00498\ldots)$ is a possible choice.

**Q9.** Show that $x \in A^c$ iff

$$0 = \operatorname{dist}(x, A \setminus \{x\}) := \inf\{|a - x| : a \in A \setminus \{x\}\}.$$

**Solution.** ($\Rightarrow$) $x \in A^c$ means that for any $\epsilon > 0$, $(V_\epsilon(x) \cap A) \setminus \{x\} \neq \varnothing$. That is to say, for any $\epsilon > 0$, there is $a \in A \setminus \{x\}$ with $|x - a| < \epsilon$. So $\operatorname{dist}(x, A \setminus \{x\}) < \epsilon$ for all $\epsilon > 0$. Hence $\operatorname{dist}(x, A \setminus \{x\}) = 0$.

($\Leftarrow$) If $0 = \inf\{|a-x| : a \in A \setminus \{x\}\}$. Then for any $\epsilon > 0$, $\epsilon$ is not a lower bound of the set $\{|a-x| : a \in A \setminus \{x\}\}$, so there must be some $a \in A \setminus \{x\}$ with $|a-x| < \epsilon$. Hence $x \in A^c$.

**Q10.** Let $A = [0, \sqrt{2}) \cap \mathbb{Q}$ and let $f(x) = \mathrm{dist}(x, A \setminus \{x\})$ for $x \in \mathbb{R}$. Express $f(x)$ explicitly and so determine $A^c$.

**Solution.** Check

$$f(x) = \begin{cases} -x\,, & \text{if } -\infty < x < 0 \\ 0\,, & \text{if } 0 \le x \le \sqrt{2} \\ x - \sqrt{2}\,, & \text{if } \sqrt{2} < x < \infty \end{cases}$$

Hence $A^c = [0, \sqrt{2}]$.

**Remark.** Given any non-empty $B \subset \mathbb{R}$, $x \mapsto \mathrm{dist}(x, B)$ is always continuous.

**Q11.** Find $\delta > 0$ such that 2 is of (strictly) positive distance to the $\delta$-neighbourhood $V_\delta(3)$ of 3. Why $\delta = 1$ cannot do the job? Show that

$$\lim_{x \to 3} \frac{x^2+1}{x-2} = 10.$$

**Solution.** Pick $\delta = 1/2$, then $\mathrm{dist}(2, V_\delta(3)) = 1/2 > 0$. $\delta = 1$ cannot do the job because $\mathrm{dist}(2, V_\delta(3)) = 0$.

For the limit, observe that

$$|\frac{x^2+1}{x-2} - 10| = |\frac{x^2-10x+21}{x-2}| = |\frac{(x-7)(x-3)}{x-2}|.$$

Let $\epsilon > 0$. Because $x - 3 \to 0$ as $x \to 3$, for the positive number $\epsilon\frac{1}{9}$, there is $\delta > 0$ s.t. for all $x \in V_\delta(3) \setminus \{3\}$,

$$|x-3| < \epsilon\frac{4}{9}.$$

Put $\delta' = \min(\delta, 1/2)$. For all $x \in V_{\delta'}(3) \setminus \{3\}$, we have that

- $|x-2| \ge 1/2$
- $|x-7| < \frac{9}{2}$
- $|x-3| < \epsilon\frac{1}{9}$

Hence

$$|\frac{x^2+1}{x-2} - 10| = \frac{|x-7| \cdot |x-3|}{|x-2|} < 2\frac{9}{2}\epsilon\frac{1}{9} = \epsilon.$$

$\therefore \lim_{x\to 3} \frac{x^2+1}{x-2} = 10.$

**Q12.** Let $\lim_{x \to c} g(x) = l_2 \ne 0$. Apply the def of limits to a suitable $\epsilon > 0$ for getting $\delta > 0, k > 0$ such that $|g(x)| \ge k$, $\forall x \in V_\delta(c) \cap (A \setminus \{c\})$. Why $\epsilon = |l_2| > 0$ cannot do the job?

**Solution.** Our choice of $\epsilon$ is $|l_2|/2$. By definition there is $\delta > 0$ s.t. for all $x \in V_\delta(c) \setminus \{c\}$,

$$|g(x) - l_2| < \frac{|l_2|}{2},$$

i.e.

$$-\frac{|l_2|}{2} + l_2 < g(x) < \frac{|l_2|}{2} + l_2.$$

Now we have two cases:

**Case 1.** $l_2 > 0$. Then

$$\frac{|l_2|}{2} = -\frac{|l_2|}{2} + l_2 < g(x) = |g(x)|.$$

**Case 2.** $l_2 < 0$. Then

$$-|g(x)| = g(x) < \frac{|l_2|}{2} + l_2 = -\frac{|l_2|}{2}.$$

Hence $|g(x)| \geq \frac{|l_2|}{2}$.